# *On Birational Transformations of Curves of High Genus.*

By Virgil Snyder.

The purpose of this paper is to show that nonsingular curves and others of genus exceeding a given number cannot be transformed into other curves of the same order by birational transformations other than collineations. The method employed will be two-fold; for the nonsingular curves we consider sections of a certain ruled surface then establish it for higher cases by induction. For the other cases, the "$n$-gonal" series of Bertini and certain inequalities will be employed in connection with linear transformations of hyperspace.

1. Given two nonsingular plane curves of order four, $c_4$, $c_4'$ in $(1, 1)$ point correspondence such that $A$, $A'$ and $B$, $B'$ are two pairs of corresponding points. Find $c_4''$, projective with $c_4'$, such that $A'' \equiv A$, $B'' \equiv B$, but otherwise unrestricted. Turn the plane of $c_4''$ about $AB$ through any angle. The lines joining pairs of corresponding points $P$, $P''$ will generate a ruled surface of order 6 and genus 3. But by Wiman's formula*

$$p \leqq \tfrac{1}{6}(n-2)(n-3)$$

wherein $p$ is the genus, $n$ the order of a ruled surface not contained in a linear congruence. Hence the sextic must belong to a linear congruence and the only quartic curves upon it lie in the planes of the pencil whose axis is a double generator. From the method in which the surface was generated, the points $C$, $D$ in which $AB$ cuts $c_4$ and $C''$, $D''$ in which it cuts $c_4''$ must be corresponding points no matter what points were chosen for $A$, $B$. Hence $c_4$, $c_4''$ are projective.†

For nonsingular $c_5$ and $c_6$ exactly the same reasoning can be employed; for $n \geqq 7$, however, no new results are obtained, since the resulting ruled surfaces are of too high an order to preclude the possibility of the required genus.

* A. Wiman, "Klassification af regelytorna af sjette graden." Dissertation, Lund, 1892. See also *Acta Mathematica*, vol. 19 (1893).

† This proof for $c_4$ was given by me in the Journal, vol. 25 (1904), p. 187, but only too brief an outline of its extension to higher orders was there given.

2. From the Brill-Noether theorem we know that if any $c_n$ is birationally transformed into $c_N$, the adjoint curves $\phi_{n-3}$ go over into $\phi_{N-3}$, and conversely every such transformation which transforms the entire system $\phi_{n-3}$ into a system $\phi_{N-3}$ will transform $c_n$ into some $c_N$. From what we have just seen, no transformation except collineations can transform the $\infty^2 c_1$, the $\infty^5 c_2$ or the $\infty^9 c_3$ into themselves.

If a $c_7$ of $p = 15$ can be transformed into itself or any other $c_7$, the $\infty^{14}$ system of $\phi_4$ must also remain invariant; but this system may be defined by fourteen nonsingular curves of the system, which must be linearly transformed.

$$\phi_4 = \Sigma \, \phi_i \, . \, a_i .$$

From § 1 this is only possible by collineations. In the same manner for $c_8$, since we can define the adjoint system by nonsingular $\phi_5$, and for $c_9$, $\phi_6$. Since the theorem is true for $c_\kappa$, $c_{\kappa+1}$, $c_{\kappa+2}$ by repeating this process, it is true for all nonsingular curves, hence:

*When a* (1, 1) *correspondence can be established between the points of any two nonsingular plane curves, they are projectively equivalent.**

3. In case of $c_5$ of $p = 5$, $\phi_{n-3}$ are conics passing through the node. If this point be (0, 0, 1), the equation of the system may be written

$$a x^2 + 2 h x y + b y^2 + 2 g x z + 2 f y z = 0.$$

It must go into itself by all the birational transformations of $c_5$. If we put

$$\rho x_1 = x^2, \; \rho x_2 = x y, \; \rho x_3 = y^2, \; \rho x_4 = x z, \; \rho x_5 = y z$$

and regard $x_i$ as homogeneous point coordinates in a linear space of four dimensions $R_4$, the image of $c_5$ is a $c_8$, whose intersections with $R_3$ are the eight points common to

$$x_2^2 = x_1 x_3 , \; x_3 x_4 = x_2 x_5 , \; x_1 x_5 = x_2 x_4 . \dagger$$

* This theorem presupposes $n > 3$. Two cubic curves in (1, 1) correspondence are projectively equivalent as a whole, though any nonsingular $c_3$ can be birationally transformed into itself by an infinite number of non-linear transformations. For $n = 4$ the result immediately follows from consideration of the adjoint curves, which are straight lines. The theorem is stated without proof for $n = 5$ and $n = 6$ by Wiman, "Ueber die algebraischen Kurven von den Geschlechtern $p = 4$, 5, 6, welche eindeutige Transformationen in sich besitzen," *Stockholm Akademien Handlingar*, Bihang, vol. 21, no. 3, pp. 1–43 (1895). The general theorem is similarly stated by me in the JOURNAL, *l. c.* and also by C. Küpper, "Ueber das Vorkommen von linearen Schaaren $g_n^2$ auf Kurven $n^{\text{ter}}$ Ordnung . . .," *Prager Sitzungsberichte*, 1892, p. 264.

† While this depiction has been extensively employed by various writers, it has apparently not been used for the present purpose. The number of linearly independent quadratic relations among the $\phi$ curves was found by Weber, "Ueber gewisse in der Theorie der Abelschen Funktionen auftretende Ausnahmefälle," *Math. Ann.* vol. 13 (1877). Other particular cases were discussed by Kraus in *Math. Ann.* vol. 16.

But not every curve of genus 5 can be reduced to a nodal quintic, hence we should expect certain relations in consequence of the reduced number of moduli.*

In this case we have

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_4}{x_5},$$

hence the quadrics have a ruled hypersurface in common, having $x_1=x_2=x_3$ for directrix, and generators of the form $x_1=\lambda x_2$, $x_2=\lambda x_3$, $x_4=\lambda x_5$. The directrix is the image of the node, and the generators are the images of the straight lines passing through it. Every linear transformation in $R_4$ which leaves the system of quadrics invariant must also transform this ruled hypersurface into itself; incidentally, therefore the directrix must remain fixed and the generators can only be permuted among themselves. The node (0, 0, 1) and the pencil $x=\lambda y$ of our plane $c_5$ therefore go into themselves. Among the $\infty^4$ adjoint conics are $\infty^3$ degraded ones, consisting of a line through the node and any other line, which must go into themselves; that is: *A nodal plane quintic curve can be transformed into itself or any other nodal quintic only by collineations.*

4. Exactly the same method will apply to curves of any order greater than 4 and having a single node. A number of the quadrics in $R_n$ will have an invariant configuration in common which is the image of the node and the pencil of straight lines through it. As the $\infty^{p-1}$ system of $\phi_{n-3}$ can be transformed into itself by collineations only (by § 1), we may say: *When two curves* ($p>2$) *having a single node are in* (1, 1) *point correspondence, they are projectively equivalent.*

5. Moreover, we can draw the same conclusions from curves of order $n$ and having a single multiple point $P_i$ of order $2\leqq i\leqq n-3$. In case $i=n-3$, $p=2n-5$ and the $(2n-7)(n-4)$ quadrics have a ruled hypersurface in common, whose $(n-3)$ fold directrix is the image of $P_i$ and the generators are the images of the lines through it. The adjoint $\phi_{n-3}$ have an $(n-2)$ fold point, hence the equation is of the form

$$u_{n-3}(x,y)\,z+u_{n-2}(x,y)=0.$$

The only transformation that will transform this system into itself is a collineation.†

* The normal curves for general moduli are given for every value of $p$ in Clebsch-Lindemann's Geometrie, vol. 1, p. 709.

† See Jung, "Ricerche sui sistemi lineari di curve algebriche di genere qualunque," *Ann. di Mat.* vol. 15 (1887), and vol. 16 (1888). This theorem has been employed by Kantor and by Wiman in their enumeration of finite groups of Cremona transformations.

*If two plane curves of order $n$ have a single multiple point of order $i \leqq n - 3$, they can have no $(1, 1)$ transformation into each other except by collineation.*

Such curves have a linear series $g^1_\kappa$, and when $i = n - 3$, they are particular cases of Bertini's trigonal curves.*

Moreover, this result can also be derived from Zeuthen's formula

$$c - c' = 2e'(p-1) - 2e(p'-1).$$

Here $e = 3$, $e' = 1$, $c' = 0$, $p' = 0$, hence $c = 2p + 4 = 4n - 6$ for the number of curves cutting $g^1_3$ which touch the given curve; but this is exactly the number of tangents to $c_n$ from $P_{n-3}$. In every case the three points of each group of $g^1_3$ are collinear, and the lines are always concurrent if $p = 2m - 5$. If $p > 4$, evidently $c_n$ cannot have two $g^1_3$, no matter what its configuration of double points may be, for if $\phi + \mu\phi' = 0$, $\psi + \lambda\psi' = 0$ be two systems of adjoints of order $n - 4$, between $\mu$, $\lambda$ would exist a $(3, 3)$ relation, but such a correspondence has a maximum genus 4. No curve of order greater than 4 can have general moduli and possess a $g^1_3$.

6. A $c_6$ having two nodes has a system of $\infty^7$ $c_3$ with the nodes for common basis points for adjoint curves. Among these curves are $\infty^4$ which factor into a line through each node and one other line. The pencils through the nodes must either remain invariant or simply interchange, hence the third line must go into a line.

*A binodal sextic curve cannot be transformed into another binodal sextic except by collineation.*

By means of §§ 1, 2 we can conclude in general: *When between the points of two binodal curves $(p > 4)$ a $(1, 1)$ correspondence exists, the curves are projectively equivalent.*

Now suppose $c_n$ has a $\varkappa$-fold point, $\varkappa \leqq n = 4$, and a double point, but no other singularities. The $g'_{n-\kappa}$ through $P_\kappa$ and the $g'_2$ through $P_2$ must both remain invariant, and the residual $\phi_{n-5}$ having a $(\varkappa - 2)$-fold point must go into a similar curve. From the last preceding case it is only possible by collineations when $n = 6$. By § 2, this is also true for $n = 7$ and $n = 8$. From three consecutive sets we may build up any case, hence: *When a $c_n$ has a $\varkappa$-fold point $(\varkappa \leqq n - 4)$ and a double point it can be transformed into another $c_n$ only by collineations.*

* Bertini, "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico," *Ann. di Mat.* vol. 22 (1894), pp. 1–40. The result of this § could be obtained from $g)$ p. 31.

7. Two fundamental questions arise from the preceding theorems: *a*) What is the largest number of double points a curve may have and not have a group of birational transformations except collineations? *b*) What is the lowest order of a curve to which a nonsingular curve may be transformed by birational but not by linear transformations? We shall answer these two questions in the order in which they are given.

8. On any curve the straight lines of the plane cut out a series $g_n^2$. Whenever it is possible to define another $g_n^2$, whose groups are not on lines, the curve can be transformed into another curve of the same order.

If $\delta$ be the largest number of double points such a curve can have, $\delta$ is less than the minimum for a space curve of the order $n$, since the latter has $g_n^3$ formed by the $\infty^3$ planes of space. By projecting upon a plane, only those groups will be collinear points whose planes pass through the center of projection. Hence every such curve will have proper $g_n^2$. First then

$$p > \tfrac{1}{2}(n-1)(n-2) - \left[\left(\frac{n-1}{2}\right)^2\right]$$

For such values of $p$, no $g_n^3$ exist, hence $g_n^2$ are complete series. We are only concerned with $n > 6$, hence $g_n^2$ is a special series, and can therefore always be cut from $c_n$ by a proper or composite $\phi_{n-3}$. If one such $G_n$ be given, and a $\phi_{n-3}$ be passed through it, the residual points of intersection of $\phi_{n-3}$, $c_n$ form the basis points of $\infty^2$ such $\phi_{n-3}$; the variable intersections then constitute the $g_n^2$ to which the given $G_n$ belongs (Riemann-Roch theorem). It was shown by Kantor* that if these $\infty^2$ $\phi_{n-3}$ are all composite, they must consist of one fixed curve and a variable curve of order $x$, the latter constituting an irreducible net.

Let $d$ double points of $c_n$ be among the basis points of the net. Of the $nx$ intersections of $c_n$, $c_x$, $n$ are to be variable and $d$ lie in the double points. The number of fixed basis points of the net is therefore $n(x-1)-d$. It has been shown by Küpper† that the maximum number of fixed basis points of $\infty^2$ $c_x$ is $x^2-(x-1)$, hence

$$d \geqq (x-1)(n-x)-1$$

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*S. Kantor, "Neue Theorie der eindeutigen periodischen Transformationen in der Ebene," *Acta Math.* vol. 19 (1895), pp. 115–193.

† C. Küpper, in the *Prager Abhandlungen*, series 7, vol. 3 (1889).

*The $c_x$ which determine $g_n^2$ on $c_n$ must pass through at least* $(x-1)(n-x)-1$ *double points on $c_n$.*

9. It is therefore necessary to take for $x$ the value which will make $d$ a minimum, provided such a curve is possible. A few illustrations will be given.

If $n=6$, $\delta=3$. Evidently any trinodal $c_6$ can be transformed into a $c_6$ by quadric inversion, using the nodes for fundamental points. No such conic is possible, however, if the three nodes are collinear, or coincident (§ 5).

If $n=7$, $\delta=7$. Evidently $\phi_{n-4}$ are here the $\infty^2$ $c_3$ through the seven nodes. If the double points are coincident, $\delta<7$. $c_7$ with $2P_3$ or $P_3+2P_2$ can be transformed into similar $c_7$ by quadratic inversion. If a $P_4$ be present, no $g_x^2$ exists unless a double also exists, i. e. $\delta=7$.

If $n>7$, $x>\frac{n-1}{2}$, since $c_x$ must contain at least $(x-1)(n-x)-1$ double points of $c_n$ among its fixed basis points. When $n=8$, $x=4$, and $\delta=11$. These points can be assumed at will, but the resulting curve is a particular one of order 8 and genus 10.

If $n=10$, $x$ may be 7, 6, 5. The corresponding values of $\delta$ are 17, 19, 19, but $\delta=17$ leads to a contradiction. Through $17P_2$ and a $G_{10}$ we can pass a $c_6$, but when $x=6$, $\delta=19$. If $c_{10}$ has $P_4+P_5$ $(\delta=16)$ or $P_4+2P_3$ $(\delta=12)$ the transformation is possible, but not for $P_7$, $\delta=21$ (§5). To construct a $c_{10}$ having 19 double points and a $g_{10}^2$, pass two $c_5$ through any four fixed points on $c_1$. These $c_5$ will intersect in 21 further points which are basis points of a net of $c_5$. Let 19 of these be chosen as double points on $c_{10}$; through them and five points on the same $c_1$ pass two $c_6$. We now have two pencils $c_5+\lambda c_5'=0$, $c_6+\mu c_6'=0$. Let $\lambda$, $\mu$ be so chosen that the $c_5$, $c_6$ passing through a given point will be corresponding. If the point be chosen on $c_1$, the correspondence will be (1, 1) because $c_5$ has 4 points on $c_1$ fixed, and $c_6$ has 5. The remaining locus will be $c_{10}$ having $19P_2$. The net of $c_5$ or the net of $c_6$ will each cut from it a $g_{10}^2$.

10. Since $\frac{n}{2}\leqq x\leqq n-3$, and $\delta$ is small for larger values of $x$, it easily follows that *the largest number of distinct double points which a curve of order $n$ may have without being birationally transformable into another curve of the same order is two less than the minimum number of double points which a space curve of the same order can have.* $(n>7)$.

11. For large values of $n$, only very particular curves $c_n$ can be so transformed. To obtain a more precise limit, express the condition that a $G_n$ cannot lie on $c_{x-1}$.

$$(x-1)(n-x)-1+x \leqq \tfrac{1}{2}(x-1)(x+2).*$$

Hence $x < \frac{2(n-1)}{3}$, from which we can say:

*A $c_n$ containing a $g_n^2$ not lying on the $\infty^2$ lines of the plane cannot have less than $(x-1)(n-x)-1$ double points, $x$ being the largest integer less than $\frac{2(n-1)}{3}$.*

Thus for $n=11$, $\delta=24$. It is an interesting exercise to construct this $c_{11}$. The double points cannot be assumed at will; they are among the 31 intersections of two sextics having 5 points on a straight line in common. A $c_{11}$ with $2\,P_5$ $(\delta=20)$ can be transformed into a similar $c_{11}$ by inversion.

12. We now consider question ($b$). A general curve of genus $p$ has $3p-3$ moduli which are undisturbed by birational transformation, but the most general $c_n$ has not more than $\frac{n(n+3)}{2}-8$ or $p+3(n-3)$ such moduli. Any non-singular $c_n$ can by quadratic inversion be transformed into a $c_{2n-3}$, having three $(n-2)$-fold points and no other singularities, and conversely.

Given two curves, $c_x$, $c_y$, $x \leqq y$. Their $xy$ intersections are such that every $c_{x+y-3}$ through all but one of them will contain that one also (Cayley). It is a minimum group on the curve. If $x+y=m+3$, then $x \leqq \frac{m+3}{2}$. The number of points in the group varies from $m+2$ to $\left(\frac{m+3}{2}\right)^2$ if $m$ is odd, or to $\frac{(m+2)}{2}\cdot\frac{(m+4)}{2}$ if $m$ is even. Our problem may now be stated thus: to find the smallest value of $\kappa$ for which $g_\kappa^2$ exists on $c_n$, $\kappa > n$. It has been partially treated by Küpper† and his definitions will be here reproduced.

$G_Q$ is called normal as to $c_m$ if any $c_m$ through $Q-1$ of the points does not have to pass through the other also, otherwise $G_Q$ is abnormal as to $c_m$.

If $G_Q$ is determined by $Q-q$ conditions, $q$ is called the excess of $G$ as to $c_m$.

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* It was shown by Küpper, *Prager Berichte* (1892), that if $c_g$ contains a $G_n$ of $g_n^2$ it imposes exactly $g+1$ conditions. This inequality is then proved, but a large number of errors have made the application there made of it of small value.

* C. Küpper, "Zur Theorie der algebraischen Curven," *Monatshefte für Math. und Physik*, vol. 6 (1895), pp. 127–156.

By the Riemann-Roch theorem, $q$ is the number of degrees of freedom in the series of curves having the residual of $G_Q$ for basis points. If $G_Q$ is abnormal as to $c_m$, but every $G_{Q-1}$ contained in it is normal, $G_Q$ is called primitive. The following theorem can now be easily proved: *If it be impossible to pass $c_i$ through $G_Q^q$, then the excess of the group as to $c_{m-i}$ is $\frac{1}{2}(i+1)(i+2)$.**

An abormal $G_Q$ containing the smallest number of points is called a minimum group as to $c_m$. We can now prove the theorem:

*Given a primitive $G_{xy}$ and $x+y=m+3$, $x \leqq y$; $G_{xy}$ is a minimum group for $c_m$ unless it lies on $c_i$ $(i<x)$.*

We first prove the following lemma: If $n$ is the lowest order a curve can have which contains $G_Q$, and $n>\frac{m+3}{2}$ say $n=\frac{m+3+\delta}{2}$, then $Q>\left(\frac{m+3}{2}\right)^2$.

First, $2n=m+3+\delta$ or $n-1=m-(n-2-\delta)$.

On putting $n-2-\delta=i$, the excess of $G_Q$ as to $c_{n-1}$ is $\geqq \frac{(n-1-\delta)(n-\delta)}{2}$. If now $c_{n-1}$ through $G_Q$ is impossible

$$\frac{(n-1)(n+2)}{2}+\frac{(n-1-\delta)(n-\delta)}{2}-Q<0,$$

so that

$$Q>n(n-\delta)+\frac{\delta}{2}(\delta+1)-1$$

or

$$Q \geqq \left(\frac{m+3}{2}\right)^2-\frac{\delta^2}{4}+\frac{\delta}{2}(\delta+1),$$

hence

$$Q>\left(\frac{m+3}{2}\right)^2 \text{ unless } \delta=0.$$

It follows therefore that $z>\frac{m+3}{2}$ could not be the lowest order of a curve through the group, nor $x<z \leqq \frac{m+3}{2}$, for such a minimum group would consist of $z(m+3-z)$ or more than $xy$ points. Since $i<x$ was supposed impossible, $c_x$ is the curve of lowest order through the group. If $Q<xy$ and $G_Q$ primitive, $c_i\,(i<x)$ can be passed through it. If $Q<2(m+1)$, $G_Q$ lies on a straight line; if $Q<3m$, $G_Q$ lies on a straight line or a conic.

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* Küpper, *l. c.*

13. Consider $c_5$. We have seen that no $g_5^2$ lies upon it except that defined by straight lines. If a $g_6^2$ be possible, it must be a special series, hence each $G_6$ lies on $\phi_2$. Pass a $c_2$ through a $G_6$. The residual is a $G_4$, through which $\infty^2$ conics should be possible, but this is only possible when all four points are collinear. The $\infty^2$ lines would then have to cut $c_5$ in 6 points, but this is impossible if $c_5$ is irreducible. *A non-singular quintic curve cannot be transformed into a sextic by any birational transformation.* Since $7 = 2n - 3$, the $\infty^2 c_2$ through three arbitrary points on $c_5$ will define a $g_7^2$. In exactly the same way it can be shown that a $c_6$ of $p = 10$ cannot be transformed into a $c_7$ nor $c_8$.

14. A $g_Q^q$ is special if $q > Q - p + 1$. Hence if $q = 2$, and $p = \frac{1}{2}(n-1)(n-2)$, the group $G_Q^2$ will be special for every value of $\varkappa < n - 3$ when $Q = \varkappa n - \beta$, $\beta < n$. We need consider only $\varkappa = 2$ and $\beta \geqq 3$, in which case $G_{2n-\beta}^2$ is always defined by $\phi_{n-3}$. But these adjoint $\phi_{n-3}$ are composite, and the variable curve is of order $\varkappa$ or $\varkappa - 1$, hence all the $g_{2n-\beta}^2$ can be cut from $c_n$ by $c_2$, and hence $\beta = 3$. *In general a non-singular $c_n$ cannot be birationally transformed into a curve of order lower than $2n - 3$. It can always be transformed into a $c_{2n-3}$ by means of quadric inversion, which is birational for the entire plane.*

This method will also apply to singular curves with special moduli, but for complicated $P_i$ the number of particular cases becomes very large.

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